

HEAT KERNEL ASYMPTOTICS ON SYMMETRIC SPACES

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Abstract. We develop a new method for the calculation of the heat trace asymptotics of the Laplacian on symmetric spaces that is based on a representation of the heat semigroup in form of an average over the Lie group of isometries and obtain a generating function for the whole sequence of all heat invariants.

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1 Introduction

The heat kernel is one of the most powerful tools in mathematical physics and geometric analysis (see, for example the books [27, 20, 28, 17, 29] and reviews [2, 22, 13, 16, 19, 33]). In particular, it is widely used to study the propagators of quantum fields and the effective action in quantum field theory and the correlation functions and the partition function in statistical physics. The short-time asymptotic expansion of the trace of the heat kernel determines the so-called spectral invariants of the differential operator in question which is intimately related to the renormalization of quantum field theories [22, 17, 33, 29], the high-temperature expansion in statistical physics [28], the dynamics of integrable systems, in particular, the Korteweg-de Vries hierarchy [28, 18], as well as spectral geometry and index theorems [27].

There has been a tremendous progress in the explicit calculation of spectral asymptotics in the last thirty years [26, 2, 3, 4, 6, 32, 35] (see also [27, 16, 19, 33, 29] and

the references therein). However, due to the combinatorial explosion in the complexity of the spectral invariants further progress in the “brute force” approach is unlikely, even if employing computer software for symbolic calculations. The results are so complicated that it requires about twenty pages to describe them [32, 35]. It seems that the further progress in the study of spectral asymptotics can be achieved by restricting oneself to operators and manifolds with high level of symmetry, for example, homogeneous and symmetric spaces, which enables one to employ powerful algebraic methods. In some very special particular cases, such as group manifolds, spheres, rank-one symmetric spaces, split-rank symmetric spaces etc, it is possible to determine the spectrum of the differential operator exactly and to obtain closed formulas for the heat kernel in terms of the root vectors and their multiplicities [1, 22, 23, 24, 28, 25]. The complexity of the method crucially depends on the global structure of the symmetric space, most importantly its rank. Most of the results for symmetric spaces are obtained for rank-one symmetric spaces only. However, it is well known that the spectral asymptotics are determined essentially by local geometry. They are polynomial invariants in the curvature with universal constants that do not depend on the global properties of the manifold. It is *this universal structure that we are interested in this paper*. We will report on our results obtained in the paper [12] (see also [8]). Related problems in a more general context are discussed in [9, 11, 14].

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2 Spectral Asymptotics

2.1 Laplacian

Let (M, g) be a smooth n -dimensional compact Riemannian manifold without boundary with a positive definite Riemannian metric g . Let TM and T^*M be the tangent and cotangent bundles of the manifold M . The metric g defines in a natural way the Riemannian volume element $d\text{vol}$ on M . Let $C^\infty(M)$ be the space of smooth real-valued functions on M . Using the invariant Riemannian volume element $d\text{vol}$ on M we define the natural L^2 inner product in $C^\infty(M)$, and the Hilbert space $L^2(M)$ as the completion of $C^\infty(M)$ in the corresponding norm.

The metric also defines the torsion-free compatible connection ∇^{TM} on the tangent bundle TM , so called Levi-Civita connection. Using the Levi-Civita connection we naturally obtain connections on all bundles in the tensor algebra over TM and T^*M ; the resulting connection will be denoted just by ∇ . It is usually clear which bundle's connection is being referred to, from the nature of the section being acted upon. With our notation, Greek indices, μ, ν, \dots , label the local coordinates $x = (x^\mu)$ on M and range from 1 through n . Let ∂_μ be a coordinate basis for the tangent space $T_x M$ at a point $x \in M$ and dx^μ be dual basis for the cotangent space $T_x^* M$. We adopt the notation that the Greek indices label the tensor components with respect to local coordinate frame and range from 1 through n . We also adopt the Einstein convention and sum over repeated indices.

Let $\nabla^* : C^\infty(T^*M) \rightarrow C^\infty(M)$ be the formal adjoint of $\nabla : C^\infty(M) \rightarrow C^\infty(T^*M)$ with respect to the L^2 inner product. A partial differential operator $\Delta : C^\infty(M) \rightarrow C^\infty(M)$ of the form

$$\Delta = -\nabla^* \nabla = g^{\mu\nu} \nabla_\mu \nabla_\nu \quad (2.1)$$

is called the scalar Laplacian. In local coordinates the Laplacian is a second-order partial differential operator of the form

$$\Delta = |g|^{-1/2} \partial_\mu |g|^{1/2} g^{\mu\nu} \partial_\nu, \quad (2.2)$$

where $|g| = \det g_{\mu\nu}$. It is easy to see that the principal symbol of the operator $(-\Delta)$ is

$$\sigma_L(-\Delta; x, \xi) = g^{\mu\nu}(x) \xi_\mu \xi_\nu, \quad (2.3)$$

where $\xi \in T_x^* M$ is a covector.

2.2 Heat Kernel

The following is known about the operator $(-\Delta)$ [27]. First of all, it is elliptic and self-adjoint. More precisely, it is essentially self-adjoint, i.e. it has a unique self-adjoint extension to $L^2(M)$. Second, the operator $(-\Delta)$ has a positive definite leading symbol. The spectrum of such operators forms a real nondecreasing sequence, $\{\lambda_k\}_{k=1}^\infty$, with each eigenspace being finite-dimensional. The eigenvectors $\{\varphi_k\}_{k=1}^\infty$ are smooth functions and form an orthonormal basis in $L^2(M)$. Moreover, as $k \rightarrow \infty$ the eigenvalues increase as $\lambda_k \sim Ck^2$ as $k \rightarrow \infty$, with some positive constant C . These facts lead to the existence of spectral asymptotics that will be discussed below.

The heat semi-group is a one-parameter family of bounded operators $U(t) = \exp(t\Delta) : L^2(M) \rightarrow L^2(M)$ for $t > 0$. The integral kernel $U(t|x, x')$ of this operator, called the heat kernel, is defined by

$$U(t|x, x') = \sum_{k=1}^\infty e^{-t\lambda_k} \varphi_k(x) \varphi_k(x'), \quad (2.4)$$

where each eigenvalue is counted with multiplicities. It satisfies the heat equation

$$(\partial_t - \Delta)U(t|x, x') = 0 \quad (2.5)$$

with the initial condition

$$U(0^+|x, x') = \delta(x, x'), \quad (2.6)$$

where $\delta(x, x')$ is the Dirac delta-function. For $t > 0$ the heat kernel is a smooth function on $M \times M$ with a well defined diagonal

$$U^{\text{diag}}(t|x) = U(t|x, x). \quad (2.7)$$

2.3 Spectral Invariants

For any $t > 0$ the heat semi-group $U(t) = \exp(t\Delta)$ is a trace-class operator with a well defined L^2 trace, called the heat trace,

$$\text{Tr}_{L^2} \exp(t\Delta) = \int_M d\text{vol} U^{\text{diag}}(t) = \sum_{k=1}^\infty e^{-t\lambda_k}. \quad (2.8)$$

The heat trace is obviously a spectral invariant of the operator $(-\Delta)$. It determines other spectral functions by integral transforms. In particular, the zeta-function, $\zeta(s, \lambda)$,

is defined as the L^2 trace of the complex power of the operator $(-\Delta - \lambda)$,

$$\zeta(s, \lambda) = \text{Tr}_{L^2}(-\Delta - \lambda)^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty dt t^{s-1} e^{t\lambda} \text{Tr}_{L^2} \exp(t\Delta), \quad (2.9)$$

where s and λ are complex variables with $\text{Re } \lambda < \lambda_1$ and $\text{Re } s > n/2$.

The zeta function enables one to define, in particular, the regularized determinant of the operator $(-\Delta - \lambda)$,

$$\text{Det}_{L^2}(-\Delta - \lambda) = \exp \left\{ -\frac{\partial}{\partial s} \zeta(s, \lambda) \Big|_{s=0} \right\}, \quad (2.10)$$

which determines the one-loop effective action in quantum field theory.

2.4 Asymptotic Expansion

It is well known that the heat kernel diagonal has the following asymptotic expansion as $t \rightarrow 0^+$ [27]

$$U^{\text{diag}}(t) \sim \sum_{k=0}^\infty t^k a_k. \quad (2.11)$$

The coefficients a_k are called local heat kernel coefficients. They are scalar polynomials in the curvature and its covariant derivatives which are known explicitly up to a_5 . In particular,

$$\begin{aligned} a_0 &= 1, \\ a_1 &= \frac{1}{6}R, \\ a_2 &= \frac{1}{30}\Delta R + \frac{1}{72}R^2 - \frac{1}{180}R_{\mu\nu}R^{\mu\nu} + \frac{1}{180}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}, \end{aligned} \quad (2.12)$$

where $R_{\mu\nu\alpha\beta}$ is the Riemann tensor, $R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu}$ is the Ricci tensor and $R = R^{\mu\nu}{}_{\mu\nu}$ is the scalar curvature. The coefficient a_3 was computed in [26]. The coefficient a_4 was first computed in [2] and published in [3, 4, 6] (see also [17]). The coefficient a_5 was computed in [32, 35].

The asymptotic expansion of the heat kernel diagonal can be integrated over the manifold to give the asymptotic expansion of the heat trace as $t \rightarrow 0$

$$\text{Tr}_{L^2} \exp(t\Delta) \sim (4\pi t)^{-n/2} \sum_{k=0}^\infty t^k A_k, \quad (2.13)$$

where

$$A_k = \int_M d\text{vol} a_k. \quad (2.14)$$

This is the famous Minakshisundaram-Pleijel asymptotic expansion. The coefficients A_k are spectral invariants of the Laplacian. They are often called global heat kernel coefficients or Hadamard-Minakshisundaram-De Witt-Seeley (HMDS) coefficients. This expansion is of great importance in differential geometry, spectral geometry, quantum field theory and other areas of mathematical physics, such as the theory of Huygens' principle, heat kernel proofs of the index theorems, Korteweg-De Vries hierarchy, Brownian motion etc. (see, for example, [28]).

The general structure of the heat kernel coefficients can be described as follows [6, 16, 17, 19]. We define symmetric tensors $K_{(j)}$ of type $(2, j)$ (that we call symmetric jets of order $(j-2)$) as symmetrized covariant derivatives of the curvature

$$K_{(j)}^{\alpha\beta}{}_{\mu_1 \dots \mu_j} = \nabla_{(\mu_1} \dots \nabla_{\mu_{j-2}} R^\alpha{}_{\mu_{j-1}}{}^\beta{}_{\mu_j)}, \quad (2.15)$$

where the parenthesis denote the complete symmetrization over all indices included. Next, we define all possible scalar (orthogonal) invariants of the form

$$J_{m, \mathbf{n}}^A = \text{tr}_g^A K_{(n_1+2)} \otimes \dots \otimes K_{(n_m+2)}, \quad (2.16)$$

where $\mathbf{n} = (n_1, \dots, n_m)$ is a multiindex, $|\mathbf{n}| = n_1 + \dots + n_m$, and tr_g denotes contraction with the metric to get a scalar. The index A labels all possible contractions. All invariants $J_{m, \mathbf{n}}^A$ have m curvatures and $|\mathbf{n}|$ derivatives of the curvatures. The classification of these invariants is a separate interesting problem.

Then the local heat kernel coefficients have the grading according to the number of derivatives, that is,

$$a_k = \sum_{m=1}^k a_{k,m}, \quad (2.17)$$

where

$$a_{k,m} = \sum_{\mathbf{n} \geq 0; |\mathbf{n}|=2k-2m} \sum_A C_{\mathbf{n}}^A J_{m, \mathbf{n}}^A, \quad (2.18)$$

where $C_{\mathbf{n}}^A$ are some universal constants. Notice that the dimension of the space of invariants $J_{m, \mathbf{n}}^A$ grows as a factorial for large k . This makes a direct explicit calculation of the heat kernel coefficients for large k meaningless.

The leading terms in the heat kernel coefficients with the highest number of derivatives were computed in [2, 5, 6, 21]. They have the form

$$\begin{aligned} A_k &= \int_M d\text{vol} \frac{(-1)^k k!}{2(2k+1)!} \left\{ 2R^{\mu\nu} \Delta^{k-2} R_{\mu\nu} \right. \\ &\quad \left. + (k^2 - k - 1)R \Delta^{k-2} R + \dots \right\}, \end{aligned} \quad (2.19)$$

where the dots denote the terms with less derivatives. Notice that there are only two independent invariants.

We are interested in this paper in the opposite case, that is, the terms

$$a_{k,k} = \sum_A C_0^A \text{tr}_g^A K_{(2)} \otimes \cdots \otimes K_{(2)}, \quad (2.20)$$

with no derivatives at all. More precisely, we assume that the curvature is parallel,

$$\nabla_\mu R_{\alpha\beta\gamma\delta} = 0, \quad (2.21)$$

in other words, we restrict ourselves to locally symmetric spaces. This is a much more complicated case due to the presence of more invariants and algebraic constraints on the curvature tensor. This is an essentially non-perturbative calculation. Explicit results exist only in some particular cases.

3 Symmetric Spaces

We list below some well known facts from the theory of symmetric spaces [34, 30, 31]. A Riemannian manifold with parallel curvature is called a locally symmetric space. A complete simply connected locally symmetric space is called a globally symmetric space (or, simply, a symmetric space). A symmetric space is said to be of compact, non-compact or Euclidean type if all sectional curvatures are positive, negative or zero. A product of symmetric spaces of compact and noncompact types is called a semisimple symmetric space. A general symmetric space is a product of a Euclidean space and a semisimple symmetric space.

It should be noted that our analysis is purely local. We are looking for a universal local generating function of the curvature invariants in the category of locally symmetric spaces, that adequately reproduces the asymptotic expansion of the heat kernel diagonal. This function should give all the terms without covariant derivatives of the curvature $a_{k,k}$ in the asymptotic expansion of the heat kernel, in other words all heat kernel coefficients a_k for any locally symmetric space. It turns out to be much easier to obtain a universal generating function whose Taylor coefficients reproduce the heat kernel coefficients a_k than to compute them directly.

It is obvious that flat subspaces do not contribute to the heat kernel coefficients a_k . Therefore, it is sufficient to consider only semisimple symmetric spaces. Moreover, since the coefficients a_k are polynomial in the curvatures,

one can restrict oneself only to symmetric spaces of compact type. Using the factorization property of the heat kernel and the duality between the compact and the noncompact symmetric spaces one can obtain then the results for the general case by analytic continuation. That is why we consider below only compact symmetric spaces.

3.1 Holonomy, Isometry and Isotropy Groups

Let $e_a = e_a^\mu \partial_\mu$ be a basis for the tangent space $T_x M$. Let e_μ^a be the matrix inverse to e_a^μ , defining the dual basis $\omega^a = e_\mu^a dx^\mu$ in the cotangent space $T_x^* M$. We extend these bases by parallel transport along geodesics to get local frames on the tangent bundle and the cotangent bundles, and, therefore, on any tensor bundle, that are parallel along geodesics. We adopt the notation that the Latin indices from the beginning of the alphabet, a, b, c, \dots , label the tensor components with respect to this local frame and range from 1 through n . Then the frame components of any parallel tensor (such as the curvature tensor) are constant.

The components of the curvature tensor of a compact symmetric space can be always presented in the form

$$R_{abcd} = \beta_{ik} E^i_{ab} E^k_{cd} \quad (3.1)$$

where E^i_{ab} , ($i = 1, \dots, p$), with $p \leq n(n-1)/2$ being a positive integer, is a set of antisymmetric $n \times n$ matrices and β_{ik} is some symmetric nondegenerate positive definite $p \times p$ matrix. In the following the Latin indices from the middle of the alphabet, i, j, k, \dots , will be used to denote such matrices; they should be not confused with the Latin indices from the beginning of the alphabet which denote tensor components. The Latin indices from the middle of the alphabet will be raised and lowered with the matrix β_{ik} and its inverse $(\beta^{ik}) = (\beta_{ik})^{-1}$.

Next we define the traceless $n \times n$ matrices D_i , by $(D_i)^a_b = D^a_{ib}$, where

$$D^a_{ib} = -\beta_{ik} E^k_{cb} g^{ca}. \quad (3.2)$$

The matrices D_i are known to be the generators of the holonomy algebra, \mathcal{H} , i.e. the Lie algebra of the restricted holonomy group, H ,

$$[D_i, D_k] = F^j_{ik} D_j, \quad (3.3)$$

where F^j_{ik} are the structure constants of the holonomy group. The matrix β_{ik} plays the role of the metric of the

holonomy group with the scalar curvature

$$R_H = -\frac{1}{4}\beta^{ik}F^m{}_{il}F^l{}_{km}. \quad (3.4)$$

The structure constants define the traceless $p \times p$ matrices F_i , by $(F_i)^j{}_k = F^j{}_{ik}$, which generate the adjoint representation of the holonomy algebra,

$$[F_i, F_k] = F^j{}_{ik}F_j. \quad (3.5)$$

Now, we go back to the equation (2.21). It is an overdetermined system of partial differential equations. By taking the commutator of covariant derivatives we obtain the integrability condition of this equation

$$R_{fgea}R^e{}_{bcd} - R_{fgeb}R^e{}_{acd} + R_{fgec}R^e{}_{dab} - R_{fged}R^e{}_{cab} = 0. \quad (3.6)$$

By using the decomposition of the Riemann tensor introduced above we obtain from this equation

$$E^i{}_{bc}D^c{}_{ka} - E^i{}_{ac}D^c{}_{kb} = E^j{}_{ab}F^i{}_{jk}. \quad (3.7)$$

This is the most important equation that holds only in symmetric spaces; it is this equation that makes a Riemannian manifold the symmetric space.

To explore further consequences of the equation (3.7) we introduce a new type of indices, the capital Latin indices, A, B, C, \dots , which split according to $A = (a, i)$ and run from 1 to $N = p + n$. We define a symmetric nondegenerate positive definite $N \times N$ matrix

$$(\gamma_{AB}) = \begin{pmatrix} g_{ab} & 0 \\ 0 & \beta_{ik} \end{pmatrix}. \quad (3.8)$$

This matrix and its inverse $(\gamma^{AB}) = (\gamma_{AB})^{-1}$ will be used to lower and to raise the capital Latin indices. Next, we introduce a collection of new quantities $C^A{}_{BC}$ with the nonvanishing components

$$C^i{}_{ab} = E^i{}_{ab}, \quad C^a{}_{ib} = -C^a{}_{bi} = D^a{}_{ib}, \quad C^i{}_{kl} = F^i{}_{kl}. \quad (3.9)$$

Let us also introduce rectangular $p \times n$ matrices T_a by $(T_a)^j{}_c = E^j{}_{ac}$ and the $n \times p$ matrices \bar{T}_a by $(\bar{T}_a)^b{}_i = -D^b{}_{ia}$. Then we can define $N \times N$ matrices $C_A = (C_a, C_i)$

$$C_a = \begin{pmatrix} 0 & \bar{T}_a \\ T_a & 0 \end{pmatrix}, \quad C_i = \begin{pmatrix} D_i & 0 \\ 0 & F_i \end{pmatrix}, \quad (3.10)$$

so that $(C_A)^B{}_C = C^B{}_{AC}$.

Then the matrices C_A satisfy the commutation relations

$$[C_A, C_B] = C^C{}_{AB}C_C, \quad (3.11)$$

and generate the adjoint representation of the Lie algebra \mathcal{G} of some Lie group G with the structure constants $C^A{}_{BC}$. In more details, the commutation relations have the form

$$[C_a, C_b] = E^i{}_{ab}C_i, \quad (3.12)$$

$$[C_i, C_a] = D^b{}_{ia}C_b, \quad (3.13)$$

$$[C_i, C_k] = F^j{}_{ik}C_j, \quad (3.14)$$

which makes it clear that the holonomy algebra \mathcal{H} is the subalgebra of the Lie algebra \mathcal{G} .

The matrix γ_{AB} plays the role of the metric on the group G with the scalar curvature

$$R_G = -\frac{1}{4}\gamma^{AB}C^C{}_{AD}C^D{}_{BC}. \quad (3.15)$$

It can be expressed in terms of the scalar curvature R of the symmetric space M and the scalar curvature R_H of the isotropy subgroup H

$$R_G = \frac{3}{4}R + R_H. \quad (3.16)$$

It is well known that for compact symmetric spaces the group of isometries is isomorphic to the Lie group G defined in the previous section (for more details see [34, 12]). The generators of isometries are the Killing vector fields $(\xi_A) = (P_a, L_i)$, which form the Lie algebra of isometries

$$[\xi_A, \xi_B] = C^C{}_{AB}\xi_C, \quad (3.17)$$

or

$$[P_a, P_b] = E^i{}_{ab}L_i, \quad (3.18)$$

$$[L_i, P_a] = D^b{}_{ia}P_b, \quad (3.19)$$

$$[L_i, L_k] = F^j{}_{ik}L_j. \quad (3.20)$$

The vector fields L_i form the isotropy subalgebra of the isometry algebra, which is isomorphic to the holonomy algebra \mathcal{H} . Thus, a compact symmetric space M is isomorphic to the quotient space of the isometry group by the isotropy subgroup $M = G/H$.

We will need the explicit form of the Killing vectors fields in symmetric spaces. Let us fix a point x' in the manifold M . Let $d(x, x')$ be the geodesic distance between a point x and the fixed point x' and

$$\sigma(x, x') = \frac{1}{2}[d(x, x')]^2. \quad (3.21)$$

Then the derivative $\nabla_\mu \sigma(x, x')$ is the tangent vector to the geodesic connecting the points x and x' at the point x . We let

$$y^a(x, x') = g^{ab} e_b^\mu(x) \nabla_\mu \sigma(x, x') \quad (3.22)$$

and

$$K^a_b = R^a_{cbd} y^c y^d. \quad (3.23)$$

Notice that $y^a = 0$ at $x = x'$. Then one can choose the variables y^a as new coordinates near x' and show that [12, 2, 6]

$$P_a = \left(\sqrt{K} \cot \sqrt{K} \right)^b \frac{\partial}{\partial y^b} \quad (3.24)$$

$$L_i = -D^b{}_{ia} y^a \frac{\partial}{\partial y^b}, \quad (3.25)$$

where K is a $n \times n$ matrix with the entries K^a_b .

4 Algebraic Methods for the Heat Kernel

4.1 Heat Semigroup

Let k^A be the canonical coordinates on the isometry group G so that each isometry is represented in the form $\exp \langle k, \xi \rangle$, where $\langle k, \xi \rangle = k^A \xi_A$. Then the left-invariant vector fields on the isometry group G are given by

$$X_A = X^M{}_A(k) \frac{\partial}{\partial k^M} \quad (4.1)$$

where

$$X^M{}_A(k) = \left(\frac{C(k)}{\exp C(k) - 1} \right)^M{}_A. \quad (4.2)$$

and $C(k) = k^A C_A$. The metric on the group G is given by

$$G_{MN} = \gamma_{AB} Y^A{}_M Y^B{}_N, \quad (4.3)$$

where $(Y^A{}_M) = (X^N{}_B)^{-1}$ is the inverse matrix to $X^N{}_B$. Then it is easy to see that the determinant of the metric is

$$|G| = \det G_{MN} = |\gamma| \det_G \left(\frac{\sinh[C(k)/2]}{C(k)/2} \right)^2, \quad (4.4)$$

where $|\gamma| = \det \gamma_{AB}$. Let X_2 be the Casimir operator on the group G defined by

$$X_2 = \gamma^{AB} X_A X_B. \quad (4.5)$$

Lemma 4.1. *Let $\Phi(t; k)$ be a function on the isometry group G defined by*

$$\Phi(t; k) = (4\pi t)^{-N/2} \det_G \left(\frac{\sinh[C(k)/2]}{C(k)/2} \right)^{1/2} \times \exp \left\{ -\frac{1}{4t} \langle k, \gamma k \rangle + \frac{1}{6} R_G t \right\}, \quad (4.6)$$

where $\langle k, \gamma k \rangle = \gamma_{AB} k^A k^B$. Then $\Phi(t; k)$ satisfies the heat equation

$$\partial_t \Phi = |G|^{1/2} X_2 |G|^{-1/2} \Phi, \quad (4.7)$$

and the initial condition

$$\Phi(0; k) = |\gamma|^{-1/2} \delta(k). \quad (4.8)$$

Proof. First, we notice that the function $|G|^{-1/4}$ satisfies the following equations

$$X_2 |G|^{-1/4} = \frac{1}{6} R_G |G|^{-1/4}, \quad (4.9)$$

and

$$k^A \frac{\partial}{\partial k^A} |G|^{-1/4} = \frac{1}{2} (N - X^A{}_A) |G|^{-1/4}, \quad (4.10)$$

where

$$X^A{}_A = \text{tr}_G C(k)/2 \coth[C(k)/2]. \quad (4.11)$$

By using these equations we show that eqs. (4.7) and (4.8) hold by a direct calculation.

One can show that the Laplacian on a symmetric space is simply the Casimir operator of the isometry group,

$$\Delta = \gamma^{AB} \xi_A \xi_B, \quad (4.12)$$

and, therefore, belongs to the center of the enveloping algebra, i.e.,

$$[\Delta, \xi_A] = 0. \quad (4.13)$$

Theorem 4.2. *Let*

$$\Psi(t) = \int_{\mathbb{R}^N} dk |\gamma|^{1/2} \Phi(t; k) \exp \langle k, \xi \rangle. \quad (4.14)$$

Then $\Psi(t)$ satisfies the heat equation

$$\partial_t \Psi = \Delta \Psi \quad (4.15)$$

with initial condition

$$\Psi(0) = 1, \quad (4.16)$$

and, therefore,

$$\Psi(t) = \exp(t\Delta). \quad (4.17)$$

This equation means that the formal power series as $t \rightarrow 0$ of both sides of this equation are the same.

Proof: We have

$$\partial_t \Psi(t) = \int_{\mathbb{R}^N} dk |\gamma|^{1/2} \partial_t \Phi(t; k) \exp \langle k, \xi \rangle. \quad (4.18)$$

By using the previous Lemma we obtain

$$\partial_t \Psi(t) = \int_{\mathbb{R}^N} dk |\gamma|^{1/2} \exp \langle k, \xi \rangle |G|^{1/2} X_2 |G|^{-1/2} \Phi(t; k). \quad (4.19)$$

Now, by integrating by parts we get

$$\partial_t \Psi(t) = \int_{\mathbb{R}^N} dk |\gamma|^{1/2} \Phi(t; k) X_2 \exp \langle k, \xi \rangle. \quad (4.20)$$

Next, we show that

$$X_B \exp \langle k, \xi \rangle = \xi_B \exp \langle k, \xi \rangle, \quad (4.21)$$

and, therefore,

$$X_2 \exp \langle k, \xi \rangle = \Delta \exp \langle k, \xi \rangle. \quad (4.22)$$

Thus, the function $\Psi(t)$ satisfies the eq. (4.15). The initial condition (4.16) for the function $\Psi(t)$ follows from the initial condition (4.8) for the function $\Phi(t; k)$.

4.2 Heat Kernel Diagonal

The heat kernel diagonal can be obtained by acting by the heat semigroup on the delta-function,

$$U^{\text{diag}}(t; x) = \exp(t\Delta) \delta(x, x') \Big|_{x=x'}. \quad (4.23)$$

To be able to use integral representation for the heat semigroup (4.14) obtained above we need to compute the action of the isometries $\exp \langle k, \xi \rangle$ on the delta-function.

Lemma 4.3. *Let ω^i be the canonical coordinates on the isotropy group H and $(k^A) = (q^a, \omega^i)$ be the natural splitting of the canonical coordinates on the isometry group G . Then*

$$\exp \langle k, \xi \rangle \delta(x, x') \Big|_{x=x'} = |\eta|^{-1/2} \times \det_{TM} \left(\frac{\sinh[D(\omega)/2]}{D(\omega)/2} \right)^{-1} \delta(q), \quad (4.24)$$

where $D(\omega) = \omega^i D_i$ and $|\eta| = \det g_{ab}$.

Proof. We choose the normal coordinates y^a with the origin at x' defined above and consider the equation of characteristics

$$\frac{dy^a}{ds} = \left(\sqrt{K(y)} \cot \sqrt{K(y)} \right)_b^a q^b - \omega^i D^a_{ib} y^b. \quad (4.25)$$

By expanding the right hand side in the Taylor series we get

$$\frac{dy^a}{ds} = q^a - \omega^i D^a_{ib} y^b - \frac{1}{2} \beta^{ij} g_{gb} D^a_{ic} D^g_{jd} y^c y^d q^b + O(y^3). \quad (4.26)$$

Let $f^a(s, q, \omega)$ be the solution of the equation of characteristics with the initial condition

$$f^a(0, q, \omega) = 0. \quad (4.27)$$

Up to quadratic terms we obtain

$$f^a(s, q, \omega) = \left(\frac{1 - \exp[-sD(\omega)]}{D(\omega)} \right)^a_b q^b + O(q^2) \quad (4.28)$$

In particular, we find the Jacobian

$$J(\omega) = \det \left(\frac{\partial f^a}{\partial q^b} \right)_{q=0, s=1} = \det_{TM} \left(\frac{\sinh[D(\omega)/2]}{D(\omega)/2} \right). \quad (4.29)$$

Then we have

$$\exp \langle k, \xi \rangle \delta(x, x') \Big|_{x=x'} = |\eta|^{-1/2} \delta(f(1, q, \omega)). \quad (4.30)$$

By noticing that

$$f^a(1, 0, \omega) = 0, \quad (4.31)$$

we finally obtain

$$\delta(f(1, q, \omega)) = |\eta|^{-1/2} J(\omega) \delta(q), \quad (4.32)$$

which proves the lemma.

One remark is in order here. We implicitly assumed here that $q = 0$ is the only solution of the equation

$$f^a(1, q, \omega) = 0. \quad (4.33)$$

This is not necessarily true. This is the equation of closed orbits of isometries and it has multiple solutions on compact symmetric spaces. However, these global solutions, which reflect the global topological structure of the manifold, will not affect our local analysis. In particular, they do not affect the asymptotics of the heat kernel. That is why, we have neglected them here.

Now by using the above lemmas and the theorem we can compute the heat kernel diagonal.

Corollary 4.4. *The asymptotic expansion of the heat kernel diagonal as $t \rightarrow 0$ is given by the formal asymptotic expansion of the function*

$$\begin{aligned} U^{\text{diag}}(t) &\sim (4\pi t)^{-n/2} \exp \left\{ \left(\frac{1}{8}R + \frac{1}{6}R_H \right) t \right\} \\ &\times \int_{\mathbb{R}^p} \frac{d\omega}{(4\pi)^{p/2}} |\beta|^{1/2} \exp \left\{ -\frac{1}{4} \langle \omega, \beta \omega \rangle \right\} \\ &\times \det_{\mathcal{H}} \left(\frac{\sinh [\sqrt{t} F(\omega)/2]}{\sqrt{t} F(\omega)/2} \right)^{1/2} \\ &\times \det_{TM} \left(\frac{\sinh [\sqrt{t} D(\omega)/2]}{\sqrt{t} D(\omega)/2} \right)^{-1/2}. \end{aligned} \quad (4.34)$$

Proof. First, for the splitting $(k^A) = (q^a, \omega^i)$ we have $dk = dq d\omega$. We compute the determinants of the metric (3.8)

$$|\gamma| = |\beta| |\eta|, \quad (4.35)$$

where $|\beta| = \det \beta_{ik}$. By using the equations (4.14), (4.17), and (4.24) and integrating over q we obtain the heat kernel diagonal

$$U^{\text{diag}}(t) = \int_{\mathbb{R}^p} d\omega |\beta|^{1/2} \Phi(t; 0, \omega) J(\omega), \quad (4.36)$$

where $J(\omega)$ is given by (4.29). Further, by using the eq. (3.10) we compute the determinants

$$\begin{aligned} \det_{\mathcal{G}} \left(\frac{\sinh[C(\omega)/2]}{C(\omega)/2} \right) &= \det_{TM} \left(\frac{\sinh[D(\omega)/2]}{D(\omega)/2} \right) \\ &\times \det_{\mathcal{H}} \left(\frac{\sinh[F(\omega)/2]}{F(\omega)/2} \right). \end{aligned} \quad (4.37)$$

where $F(\omega) = \omega^i F_i$. By using eqs. (4.6), (4.29) and (4.34) after scaling the integration variables $\omega \rightarrow \sqrt{t} \omega$ we obtain finally (4.34).

4.3 Heat Kernel Asymptotics

We introduce a Gaussian average over ω by

$$\langle f(\omega) \rangle = \int_{\mathbb{R}^p} \frac{d\omega}{(4\pi)^{p/2}} |\beta|^{1/2} \exp \left(-\frac{1}{4} \langle \omega, \beta \omega \rangle \right) f(\omega) \quad (4.38)$$

Then

$$\begin{aligned} U^{\text{diag}}(t) &= (4\pi t)^{-n/2} \exp \left\{ \left(\frac{1}{8}R + \frac{1}{6}R_H \right) t \right\} \\ &\times \left\langle \det_{\mathcal{H}} \left(\frac{\sinh [\sqrt{t} F(\omega)/2]}{\sqrt{t} F(\omega)/2} \right)^{1/2} \right. \\ &\left. \times \det_{TM} \left(\frac{\sinh [\sqrt{t} D(\omega)/2]}{\sqrt{t} D(\omega)/2} \right)^{-1/2} \right\rangle \end{aligned} \quad (4.39)$$

This equation can be used now to generate all heat kernel coefficients a_k for any locally symmetric space simply by expanding it in a power series in t . By using the standard Gaussian averages

$$\langle \omega_1^{i_1} \dots \omega^{i_{2k+1}} \rangle = 0, \quad (4.40)$$

$$\langle \omega^{i_1} \dots \omega^{i_{2k}} \rangle = \frac{(2k)!}{2^{2k} k!} \beta^{(i_1 i_2} \dots \beta^{i_{2k-1} i_{2k}}) \quad (4.41)$$

one can obtain now all heat kernel coefficients in terms of various contractions

$$\text{tr}_{\beta} [F \otimes \dots \otimes F \text{tr}_g (D \otimes \dots \otimes D)] \quad (4.42)$$

of the matrices $D^a{}_{ib}$ and $F j_{ik}$ with the matrices β^{ik} and g^{ab} . All these quantities are curvature invariants and can be expressed directly in terms of the Riemann tensor.

There is an alternative representation of the Gaussian average in purely algebraic terms. Let b^j and b_k^* be operators acting on a Hilbert space, called creation and annihilation operators, that satisfy the following commutation relations

$$[b^j, b_k^*] = \delta_k^j, \quad (4.43)$$

$$[b^j, b^k] = [b_j^*, b_k^*] = 0. \quad (4.44)$$

Let $|0\rangle$ be a unit vector in the Hilbert space, called the vacuum vector, that satisfies the equations

$$\langle 0|0\rangle = 1, \quad (4.45)$$

$$b^j|0\rangle = \langle 0|b_k^* = 0. \quad (4.46)$$

Then the Gaussian average is nothing but the vacuum expectation value

$$\langle f(\omega) \rangle = \langle 0|f(b) \exp \langle b^*, \beta b^* \rangle |0\rangle, \quad (4.47)$$

where $\langle b^*, \beta b^* \rangle = \beta^{jk} b_j^* b_k^*$. This should be computed by so-called normal ordering, that is, by simply commuting the operators b_j through the operators b_k^* until they hit the vacuum vector giving zero. The remaining non-zero commutation terms precisely reproduce the eqs. (4.40), (4.41).

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