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SOME OPEN PROBLEMS IN GEOMETRY

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Abstract. We shall describe some open problems in geometry that are evolved from or generated in the 2006 Midwest Geometry Conference.
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1 Introduction

We include some open problems in geometry that are evolved from or generated in "Panel discussions on open problems" held after the Saturday night banquet of the Midwest Geometry Conference in the Commons Restaurant at The University of Oklahoma. The panel is comprised of the invited speakers and participants to whom we wish to express our gratitude. Special thanks to Professors Jianguo Cao, Robert M. Hardt, Wu-Yi Hsiang, Changyou Wang, Shihshu Walter Wei, and Henry C. Wente for the following Problems 1-3, 4, 5, 6, 7-8, and 9-15 (or *Conjectures* 1-7) respectively:

1. (Liouville - type problem) Let $\Omega \subset CP^n \, (n \geq 2)$ be a compact pseudo-convex domain with smooth boundary. Then is it true

$$L^2_{(p,0)}(\Omega) \cap ker(\bar{\partial}) = \begin{cases} 0 \;\; if \;\; p \geq 0 \\ C \;\; if \;\; p = 0 \end{cases}$$

2. (Density Problem) Let $\Omega \subset CP^n \, (n \geq 2)$ be a compact pseudo-convex domain with smooth boundary. Is $L^\infty_{(p,0)}(\overline{\Omega}) \cap ker(\bar{\partial})$ dense in $L^2_{(p,0)}(\Omega) \cap ker(\bar{\partial})$?

3. (CR Prescribed Curvature Problem) Let M^{2n-1} be a compact smooth CR manifold whose Levi-form is of type (p,q) with $p+q = n-1$. Suppose that $f \in C^\infty(M)$. Can we find a metric g with scalar Webster curvature

$$W_g(x) = f(x)\,?$$

The map $\frac{x}{|x|} : B^n \to S^{n-1}$ is proved to be p-energy minimizing for $p = 2$, $n = 3$ in Brezis-Coron-Lieb [4] (with another proof in Almgren-Browder-Lieb [2]), for $p = 2 < n$ in Lin [16], $p \in \{2,3,\ldots,n-1\}$ in Gulliver-Coron [8] (with another proof in Avellaneda-Lin [3]), for $p \in (n-1,n)$ in Hardt-Lin-Wang [13], and for $p \in [2, n-2\sqrt{n-1}]$, $n > 7$ in Wang [21]. On the other hand, the map apparently has infinite p-energy for $p \geq n$. This brings up the following:

4. (Minimizing Problem of the Radial Projection) Is it true that the map $\frac{x}{|x|} : B^n \to S^{n-1}$ is p-energy minimizing

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for other ranges of p and n? For example, whether or not one can prove that the map $\frac{x}{|x|}: B^4 \to S^3$ is 2.1-energy minimizing?

5. (Global Differential Geometry Problem) What are those closed Weingarten surfaces in Euclidean 3-space?

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $g \in W^{1,\infty}(\Omega)$, recall that $u \in W_g^{1,\infty}(\Omega)$ is an *absolutely minimizing Lipschitz extension* (or AMLE) of g, if for any open set $U \subset\subset \Omega$,

(1)
$$\|\nabla u\|_{L^\infty(U)} \le \|\nabla v\|_{L^\infty(U)},\ \forall v \in W^{1,\infty}(U), v = u \text{ on } \partial U.$$

A well-known theorem by R. Jensen [14] asserts that $u \in W^{1,\infty}(\Omega)$ is an AMLE if and only if u is a viscosity solution of the infinity Laplace equation:

$$(2) \qquad \Delta_\infty u := \sum_{i,j=1}^{\infty} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i \partial x_j} = 0, \text{ in } \Omega.$$

Moreover, the Dirichlet boundary value problem, with $u|_{\partial\Omega} = g$, for both AMLE and (2) has a unique solution $u \in W^{1,\infty}(\Omega) \cap C(\overline{\Omega})$.

Aronsson [1] constructed $C^{1,\frac{1}{3}}$-viscosity solutions to the infinity Laplace equation (2). An open problem, posed by Crandall-Evans [7], asks

Problem I. *Prove that any viscosity solution of the infinity Laplace equation (2) is* $C^{1,\alpha}$ *for some* $0 < \alpha \le \frac{1}{3}$.

More recently, in a very important paper [18], Savin has proved that any viscosity solution of (2) is C^1 in dimension two. In [10], Evans-Savin prove that in dimension two, any viscosity solution is indeed $C^{1,\alpha}$. However, Problem I remains completely open for $n \ge 3$.

In general, for a uniformly convex Hamiltonian function $H \in C^2(\mathbb{R}^n)$, one can define an *absolute minimizer* (or AM) of the L^∞-functional $F(u,E) = \|H(\nabla u)\|_{L^\infty(E)}$ as follows: for any $U \subset\subset \Omega$, $F(u,U) \le F(v,U)$ for any $v \in W^{1,\infty}(U)$ with $v = u$ on ∂U. Through the works by Barron-Jensen-Wang [5], Crandall [6], Yu [26], and Gariepy-Wang-Yu [12], it is known that any AM $u \in W^{1,\infty}(\Omega)$ of $H(\cdot)$ is equivalent to a viscosity solution of Aronsson's equation:

$$(3) \qquad \Delta_\infty^{(H)} u := H_p(\nabla u) \bigotimes H_p(\nabla u) : \nabla^2 u = 0, \text{ in } \Omega.$$

Wang-Yu [20] have extended the main theorem of [18] and proved that for any uniformly convex $H \in C^2(\mathbb{R}^2)$, a viscosity solution of Aronsson's equation (3) is C^1. Inspired by Problem I, C.Y. Wang would like to pose

6. (Regularity Problem of Aronsson's Equation) *For a uniformly convex* $H \in C^2(\mathbb{R}^n)$, *is a viscosity solution of Aronsson's equation (3)* $C^{1,\alpha}$ *for some* $\alpha \in (0,1)$?

A Theorem of Gordon [11] states that a harmonic map pull back convex functions to subharmonic functions. The technique in [11] does not seem to carry over to a p-harmonic map. However, it is found in Wei-Li-Wu [24] that a horizontally weak conformal p-harmonic map (or equivalently, a p-harmonic morphism) pulls back convex functions to p-subharmonic functions. Thus, a natural question arises:

7. (p-Harmonic Map Problem) Is it true that a p-harmonic map pulls back convex functions to p-subharmonic functions?

It is proved by Sachs-Uhlenbeck [19] and Chern-Goldberg [8] that a nonconstant harmonic map on the two-sphere S^2 into a compact Riemannian manifold N of dimension > 2 is a conformal branched minimal immersion. On the other hand, in higher dimensions, it is shown in Wei [22] the Existence Theorem of nonconstant n-harmonic maps on n-sphere S^n into a compact Riemannian manifold N with $\pi_n(N) \ne 0$, generalizing the work of Sachs-Uhlenbeck [19] for $n = 2$. Furthermore, it is shown in [17] that a map between equal dimensional manifolds M^n and N^n is an n-harmonic morphism (i.e. a map that pullbacks germs of n-harmonic functions to germs of n-harmonic functions), if and only if u is weakly conformal (e.g. stereographic projections $u : R^n \to S^n$ are n-harmonic maps and n-harmonic morphisms, for all n), see [23, Example 20]. Thus, it is tempting to ask

8. (Conformal Problem) Is it true that a nonconstant n-harmonic map on the n-sphere S^n into a Riemannian manifold N of dimension greater than n $(\dim N > n)$ is a conformal map?

Some Problems (or Conjectures) with regard to Constant Mean Curvature Surfaces in $\mathbb{R}^3$

Let Σ be a constant mean curvature surface embedded in $\mathbb{R}^3$ with boundary a circle. The following results are known.

1. If Σ lies in one of the half spaces determined by the plane of the bounding circle, then Σ is a spherical cap. (The proof follows from an Alexandroff reflection argument.)

2. If Σ is transversal to the plane of the bounding circle at each boundary point of Σ then Σ is a spherical cap. (See Reference [9]).

9. (*Conjecture* 1) Suppose Σ is an embedded constant mean curvature surface whose boundary is a circle. Show that Σ is a spherical cap.

10. (*Conjecture* 2) Let Σ be an immersed constant mean curvature surface of disk type whose boundary is a circle. Show that Σ is a spherical cap.

Remark: N. Kaponleas has provided an example of an immersed constant mean curvature surface of higher genus whose boundary is a circle which is not a spherical cap. (See Reference [15]).

Definition: Let C_1, C_2 be two disjoint rectifiable closed curves in $\mathbb{R}^3$. A spanner is a constant mean curvature surface of annular type whose boundary is the pair of curves C_1, C_2.

11. (*Conjecture* 3) Let C_1, C_2 be convex closed curves lying in parallel planes in $\mathbb{R}^3$. Show that there exists a constant mean curvature spanner for C_1, C_2 (suggested to H. Wente by Harold Rosenberg).

Note: The conjecture is true if C_1, C_2 are circles.

Recall the following version of the volume constrained Plateau Problem: (See Reference [25])

Theorem(*Wente*). Let Γ be a convex closed curve lying in the *x*-*y* plane. For any given volume V_0, there exists a constant mean curvature surface Σ of disk type with boundary Γ which together with the plane of Γ encloses the volume V_0.

Note: Such surfaces exist as area minimizers among all comparison surfaces enclosing volume V_0 with boundary Γ.

Note: If Γ is a circle this area minimizer is a spherical cap and the solution is unique.

12. (*Conjecture* 4) Show that if Γ is a convex planar curve then the solution to the volume constrained Plateau Problem is unique.

13. (*Conjecture* 5) Show that there is at most one embedded constant mean curvature surface Σ with boundary a convex closed curve Γ and enclosing a volume V_0.

14. (*Conjecture* 6) Let $\Sigma(V)$ be an embedded constant mean curvature surface bounded by the convex closed curve Γ and enclosing volume V. For Γ a circle and if $\Sigma(V)$ is an area minimizer the surface is the spherical cap lying in a half-space. Also for small volumes, $\Sigma(V)$ is represented as a graph and lies in a half space. There should be a volume, V_1, such that for $V < V_1$ the area minimizer $\Sigma(V)$ will lie in a half space determined by the plane of the boundary curve Γ while for $V > V_1$ the surface $\Sigma(V)$ will no longer have this property. In other words, for large volumes the constant mean curvature surfaces $\Sigma(V)$ will spill over into the other half-space. (H. Wente learned of this conjecture from J. McCuan. Computer simulations seem to support this conjecture.)

15. (*Conjecture* 7) Let Σ_1, Σ_2 be two solutions to the volume constrained Plateau Problem so that each surface has boundary a rectifiable Jordan curve Γ enclosing the same prescribed volume V_0. If Σ_1, Σ_2 are both area minimizers, show (by some kind of min-max argument?) that there is another constant mean curvature surface, Σ_3, spanning Γ and enclosing volume V_0.

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