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DIFFERENT APPROACHES TO THE H^p BOUNDEDNESS OF RIESZ TRANSFORMS

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Abstract

We give different approaches to show the H^p boundedness of Riesz transforms for $0 < p \leq 1$.

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1 Introduction

Let ϕ be a function in $\mathcal{S}(\mathbb{R}^n)$, the Schwartz space of rapidly decreasing smooth functions, satisfying $\int_{\mathbb{R}^n} \phi(x)\, dx = 1$. Define

$$\phi_r(x) = r^{-n}\phi(x/r), \quad r > 0, \quad x \in \mathbb{R}^n,$$

and the maximal function f^* by

$$f^*(x) = \sup_{r>0} |f * \phi_r(x)|,$$

where the convolution operator "$*$" is given by

$$g * f(x) = \int_{\mathbb{R}^n} g(x-y) f(y)\, dy.$$

We say a tempered distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ is in the Hardy space $H^p(\mathbb{R}^n)$ if f^* is in $L^p(\mathbb{R}^n)$. The quasi-norm on $H^p(\mathbb{R}^n)$ is $\|f\|_{H^p}^p \equiv \|f^*\|_{L^p}^p$, which satisfies

$$\|f+g\|_{H^p}^p \leq \|f\|_{H^p}^p + \|g\|_{H^p}^p \qquad \text{for} \quad 0 < p \leq 1.$$

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When $p > 1$, H^p and L^p are essentially the same because of the celebrated theorem of Hardy and Littlewood

$$\|f^*\|_{L^p} \le C_p \|f\|_{L^p} \qquad \text{for} \quad p > 1;$$

however, when $p \le 1$ the space H^p is much better adapted to problems arising in the theory of harmonic analysis.

Let R_j, $j = 1, 2, \cdots, n$, denote the *Riesz transforms* in $\mathbb{R}^n$ defined by

$$R_j f(x) = \text{p.v. } K_j * f(x), \quad \text{where } K_j(x) = \pi^{-(n+1)/2}\Gamma\Big(\frac{n+1}{2}\Big)\frac{x_j}{|x|^{n+1}}.$$

For $n = 1$, the Riesz transforms reduce to the Hilbert transform

$$Hf(x) = \lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{|y|\ge\varepsilon}\frac{f(x-y)}{y}\,dy.$$

Use "$\hat{}$" and "$\check{}$" to denote the Fourier transform

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}\,dx$$

and its inverse transform, respectively. Then

$$\widehat{R_j f}(\xi) = -i\frac{\xi_j}{|\xi|}\hat{f}(\xi).$$

This paper is concerned with the $H^p(\mathbb{R}^n)$ boundedness of the Riesz transforms.

Using the system of conjugate harmonic functions given by Stein and Weiss [7], we have another equivalent definition of Hardy spaces as follows. We consider $n+1$ variables, $(X, y) = (x_1, x_2, \cdots, x_n, y)$, and suppose

$$F(X, y) = (u(X, y), v_1(X, y), v_2(X, y), \cdots, v_n(X, y))$$

is defined on the upper half space $\mathbb{R}^{n+1}_+ = \{(X, y) \in \mathbb{R}^n : y > 0\}$ satisfying the generalized Cauchy-Riemann equations,

$$\frac{\partial u}{\partial y} + \sum_{i=1}^n \frac{\partial v_i}{\partial x_i} = 0, \quad \frac{\partial u}{\partial x_i} = \frac{\partial v_i}{\partial y}, \quad i = 1, 2, \cdots, n,$$

$$\frac{\partial v_i}{\partial x_j} = \frac{\partial v_j}{\partial x_i} \quad i \ne j, \quad 1 \le i, j \le n.$$

These equations are assumed to hold on the upper half space $\mathbb{R}^{n+1}_+$. For $p > 0$, we say that $F \in H^p(\mathbb{R}^{n+1}_+)$ if

$$\|F\|^p_{H^p} := \sup_{y>0}\int_{\mathbb{R}^n}\Big(|u(X, y)|^2 + \sum_{i=1}^n |v_i(X, y)|^2\Big)^{p/2} dX < \infty.$$

Stein and Weiss [7, page 46] showed that if $F(X, y)$ is in $H^p(\mathbb{R}^{n+1}_+)$, $p \ge (n-1)/n$, then there exist boundary values $f(X) = F(X, 0)$ such that

$$\lim_{y\to 0} F(X, y) = f(X)$$

for almost every X in $\mathbb{R}^n$. In case $p > (n-1)/n$, $f(X)$ is also the limit in the norm of $F(X, y)$.

On the other hand, let $f \in H^1(\mathbb{R}^n)$ in the sense given in Introduction. We set $f_j = R_j(f)$, $j = 1, 2, \cdots, n$, the Riesz transforms of f, and

$$u(X,t) = P_t * f(X), \qquad v_j(X,t) = P_t * f_j(X), \quad j = 1, 2, \cdots, n,$$

where P_t is the Poisson kernel. Then,

$$F(X,t) = (u(X,t), v_1(X,t), v_2(X,t), \cdots, v_n(X,t)) \in H^1(\mathbb{R}^{n+1}_+).$$

Using the system of conjugate harmonic functions, Stein and Weiss [7] showed

Theorem 1.1. *The Riesz transforms are bounded on* $H^1(\mathbb{R}^n)$.

By using the similar approaches indicated in [7], the Riesz transforms can be extended to $H^p(\mathbb{R}^n)$ boundedness for $(n-1)/n < p \le 1$. Later on Fefferman and Stein [3, Theorem 12] extended the result to $0 < p \le 1$ by checking the kernels of Riesz transforms

$$K_j(x) = \pi^{-(n+1)/2}\Gamma\Big(\frac{n+1}{2}\Big)\frac{x_j}{|x|^{n+1}}$$

being of class C^∞ away from the origin, and

$$\Big|\Big(\frac{\partial}{\partial x}\Big)^\alpha K_j(x)\Big| \le C|x|^{-n-|\alpha|} \qquad \text{for all multi-indices } \alpha.$$

Theorem 1.2. *The Riesz transforms are bounded on* $H^p(\mathbb{R}^n)$, $0 < p \le 1$.

Next we shall give different approaches to the above result.

2 The Second Proof of Theorem 2

A bounded measurable function m defined on $\mathbb{R}^n$ is said to be an $H^p(\mathbb{R}^n)$ *multiplier*, $0 < p \le \infty$, if $f \in H^p$ implies $(m\hat{f})\check{} \in H^p$ and

$$\|(m\hat{f})\check{}\|_{H^p} \le C_p\|f\|_{H^p} \qquad (\text{with } C_p \text{ independent of } f).$$

The multiplier theorem was originally due to Hörmander [5], who considered the $L^p(\mathbb{R}^n)$ multiplier. Later on Calderón and Torchinsky [1] extended the $L^p(\mathbb{R}^n)$ multipliers to $H^p(\mathbb{R}^n)$ multipliers:

Theorem 2.1. *Let* $0 < p \le 1$ *and* $m \in C^k(\mathbb{R}^n \setminus \{0\})$, $k > n(1/p - 1/2)$. *If* $m \in L^\infty(\mathbb{R}^n)$ *and*

$$\sup_{R>0} R^{2|\alpha|-n}\int_{R<|\alpha|\le 2R}\Big|\Big(\frac{\partial}{\partial x}\Big)^\alpha m(x)\Big|^2 dx < \infty \qquad \textit{for all } |\alpha| \le k,$$

then m *is an* $H^p(\mathbb{R}^n)$ *multiplier.*

Another simple proof for Theorem 2 is to apply the above theorem and check the kernels of Riesz transforms satisfying the assumptions of Theorem 3.

Proof. [The second proof of Theorem 2] Write $m_j(x) = -ix_j/|x|$. We then have

$$\Big|\Big(\frac{\partial}{\partial x}\Big)^{\alpha} m_j(x)\Big| \le C|x|^{-|\alpha|} \qquad \text{for all multi-indices } \alpha.$$

Hence,

$$\begin{aligned} R^{2|\alpha|-n}\int_{R<|\alpha|\le 2R}\Big|\Big(\frac{\partial}{\partial x}\Big)^{\alpha} m_j(x)\Big|^2 dx &\le CR^{2|\alpha|-n}\int_{R<|\alpha|\le 2R}|x|^{-2|\alpha|}dx \\ &= CR^{2|\alpha|-n}\cdot R^{n-2|\alpha|} \\ &= C \quad \text{for all multi-indices } \alpha. \end{aligned}$$

Thus m_j satisfies the assumptions of Theorem 3, and we get the $H^p(\mathbb{R}^n)$, $0 < p \le 1$, boundedness of the mapping $f \mapsto R_j f$. ■

3 The Third Proof of Theorem 2

Let $\mathcal{D} = \mathcal{D}(\mathbb{R}^n)$ denote the subspace of $C^\infty(\mathbb{R}^n)$ of compactly supported functions with their usual topology, and $\mathcal{D}'$ denote its dual space. For $\eta \in \mathcal{D}$, $z \in \mathbb{R}^n$ and $t > 0$, let

$$\eta^{z,t}(x) = \eta\Big(\frac{x-z}{t}\Big).$$

A linear and continuous operator $T : \mathcal{D} \mapsto \mathcal{D}'$ is said to satisfy the *Weak Boundedness Property* if, for each bounded subset $\mathcal{B}$ of $\mathcal{D}$, there exists a positive constant $C = C(\mathcal{B})$ such that for all $\varphi, \psi \in \mathcal{B}$, all $z \in \mathbb{R}^n$ and all $t > 0$,

$$\big|\langle T\varphi^{z,t}, \psi^{z,t}\rangle\big| \le Ct^n.$$

In [4, 8], Frazier-Torres-Weiss and Torres studied the $T1$ theorem and used distribution theory to prove the boundedness of Calderón-Zygmund operators on the Triebel-Lizorkin spaces $\dot{F}_p^{\alpha,q}(\mathbb{R}^n)$. For the special case $\alpha = 0$ and $q = 2$, we have $\dot{F}_p^{0,2} = H^p$ for $p \le 1$. Moreover, if $k(x, y) = k(x - y)$ is the convolution type kernel, they obtained the following result.

Theorem 3.1. *[*[4, page 67] and [9, page 86] *Let* $0 < p \le 1$ *and* $[\,\cdot\,]$ *denote the integer function. If* T *is the convolution operator with kernel* k *such that*

(i) T *satisfies the Weak Boundedness Property,*

(ii) $|D^\alpha k(x)| \le C|x|^{-n-|\alpha|}$ *for* $|\alpha| \le [n(1/p - 1)]$,

(iii) $|D^\alpha k(x) - D^\alpha k(y)| \le C|x - y|^\varepsilon |x|^{-n-|\alpha|-\varepsilon}$ *for* $|\alpha| = [n(1/p - 1)]$, $2|x - y| < |x|$, *and* $1 > \varepsilon > n/p - [n/p]$,

(iv) $T(x^\alpha) = 0$ *for* $|\alpha| \le [n(1/p - 1)]$,

then T *is bounded on* $H^p(\mathbb{R}^n)$.

Proof. [The third proof of Theorem 2] Write $k_j(x) = x_j/|x|^{n+1}$. Conditions (ii) and (iii) are satisfied by direct calculations. The conditions (i) and (iv) are always satisfied by principal value convolution operators (see [9, Proposition 2.2.17 and §2.3]). Hence, the mappings $f \mapsto R_j f$ are bounded on $H^p(\mathbb{R}^n)$, $0 < p \leq 1$. ■

4 The Fourth Proof of Theorem 2

Here we present one more proof of Theorem 2. The ideas and methods come from the Coifman's atomic decomposition and Taibleson-Weiss' molecular characterization. We first give definitions of atoms and molecules, and their related results.

Definition 4.1 (Definition of atoms). Let $0 < p \leq 1 \leq q \leq \infty$, $p \neq q$, $s \in \mathbb{Z}$ and $s \geq [n(1/p - 1)]$. (Such an ordered triple (p, q, s) is called *admissible*.) A (p, q, s)-*atom* centered at $x_0 \in \mathbb{R}^n$ is a function $a \in L^q(\mathbb{R}^n)$, supported on a ball $B \subseteq \mathbb{R}^n$ with center x_0 and satisfying

(i) $\|a\|_q \leq |B|^{1/q-1/p}$,

(ii) $\int_{\mathbb{R}^n} a(x)x^\alpha\, dx = 0$ for every multi-index α with $|\alpha| \leq s$.

Let $H^{p,q,s}$ denote the space consisting of tempered distributions admitting a decomposition $f = \sum \lambda_i a_i$, where a_i's are (p, q, s)-atoms and $\sum |\lambda_i|^p < \infty$. For $f \in H^p(\mathbb{R}^n)$, we also set

$$\mathcal{N}_{p,q,s}(f) = \inf\left\{\Big(\sum_i |\lambda_i|^p\Big)^{1/p} : \sum_i \lambda_i a_i \text{ is a decomposition of } f \text{ into } (p,q,s)\text{-atoms}\right\}.$$

We have the following atomic decomposition for H^p.

Theorem 4.2. *[2, 6] If the triple (p, q, s) is admissible, then $H^p = H^{p,q,s}$. Moreover, both $\|f\|_{H^p}$ and $\mathcal{N}_{p,q,s}(f)$ are equivalent.*

If we allow an atom to have support outside of a ball and also replace its size condition, then we get a generalized atom. Such generalized atoms are called molecules and are useful in certain applications.

Definition 4.3. [Definition of molecules] Let (p, q, s) be an admissible triple and $\varepsilon > \max\{s/n, 1/p - 1\}$. (Such a quadruple (p, q, s, ε) is also called admissible.) Set $a = 1-1/p+\varepsilon$, $b = 1-1/q+\varepsilon$. A (p, q, s, ε)-*molecule* centered at x_0 is a function $M \in L^q(\mathbb{R}^n)$ satisfying

(i) $M(x) \cdot |x - x_0|^{nb} \in L^q(\mathbb{R}^n)$,

(ii) $\|M\|_q^{a/b} \cdot \big\|M(x) \cdot |x - x_0|^{nb}\big\|_q^{1-a/b} \equiv \mathfrak{N}(M) < \infty$,

($\mathfrak{N}(M)$ is called the molecular norm of M.)

(iii) $\int_{\mathbb{R}^n} M(x)x^\alpha\, dx = 0$ for every multi-index α with $|\alpha| \leq s$.

The following molecular characterization is very useful in establishing H^p boundedness of sublinear operators.

Theorem 4.4. *[8] Each (p, q, s, ε)-molecule M is in H^p and $\|M\|_{H^p} \le C\mathfrak{N}(M)$, where the constant C is independent of the molecule.*

Remark 1: Every (p, q, s)-atom f is a (p, q, s, ε)-molecule, and $\mathfrak{N}(f) \le C$ where C is a constant independent of f.

Remark 2: As a consequence of Theorems 5 and 6, for a sublinear operator T to be bounded on H^p, it suffices to show that Tf is a p-molecule and $\mathfrak{N}(Tf) \le C$ for some constant C independent of f whenever f is a p-atom.

We now prove Theorem 2 by using atom-molecule theory.

proof. [The fourth proof of Theorem 2] Since the Riesz transforms commute with translations, we consider atoms and molecules centered at the origin only. For $s \in \mathbb{N}$ and $n/(n+s) < p \le n/(n+s-1)$, we choose a number ε satisfying $1/p - 1 < \varepsilon < s/n$. Then both $(p, 2, s-1)$ and $(p, 2, s-1, \varepsilon)$ are admissible by straightforward calculations. We shall prove that if f is a $(p, 2, s-1)$-atom, then $R_j f$ are $(p, 2, s-1, \varepsilon)$-molecules with molecular norm $\mathfrak{N}(R_j f) \le C$ (C independent of f) for $1 \le j \le n$.

Given a $(p, 2, s-1)$-atom f with $\mathrm{supp}(f) \subseteq \{x \in \mathbb{R}^n : |x| \le R\}$, we have $\|f\|_2 \le C_n R^{n(1/2-1/p)}$ and $\int f(x)x^\alpha\, dx = 0$ for $0 \le |\alpha| \le s-1$. Let $a = 1 - 1/p + \varepsilon$ and $b = 1/2 + \varepsilon$. Then

$$\begin{aligned}\|R_j f(x) \cdot |x|^{nb}\|_2^2 &= \int_{\mathbb{R}^n} |K_j * f(x)|^2 \cdot |x|^{n+2n\varepsilon}\, dx \\ &= \left(\int_{|x|\le 2R} + \int_{|x|>2R}\right)|K_j * f(x)|^2 \cdot |x|^{n+2n\varepsilon}\, dx \\ &\equiv I_1 + I_2.\end{aligned}$$

The L^2 boundedness of R_j implies

$$I_1 \le (2R)^{n+2n\varepsilon}\|K_j * f\|_2^2 \le C_n R^{n+2n\varepsilon}\|f\|_2^2 \le C_n R^{2na}.$$

To estimate I_2, we use the moment condition of f to write

$$\begin{aligned}I_2 &\equiv \int_{|x|>2R}\left|\int_{|y|\le R} K_j(x-y)f(y)dy\right|^2 |x|^{n+2n\varepsilon}dx \\ &= \int_{|x|>2R}\left|\int_{|y|\le R}\left\{K_j(x-y) - \sum_{|\alpha|=0}^{s-1}\frac{1}{\alpha!}D^\alpha K_j(x)(-y)^\alpha\right\}f(y)dy\right|^2 |x|^{n+2n\varepsilon}dx.\end{aligned}$$

Taylor's theorem and Schwarz's inequality give

$$\begin{aligned}&\left|\int_{|y|\le R}\left\{K_j(x-y) - \sum_{|\alpha|=0}^{s-1}\frac{1}{\alpha!}D^\alpha K_j(x)(-y)^\alpha\right\}f(y)\, dy\right|^2 \\ &\qquad \le C_n|x|^{-2n-2s}\|f\|_2^2\int_{|y|\le R}|y|^{2s}\, dy \qquad \text{for } |x| \ge 2|y|,\end{aligned}$$

which implies

$$\begin{aligned}I_2 &\le C_n \|f\|_2^2 \int_{|y|\le R} |y|^{2s}\,dy \int_{|x|>2|y|} |x|^{2n\varepsilon-n-2s}\,dx \\ &\le C_n \|f\|_2^2 \int_{|y|\le R} |y|^{2n\varepsilon}\,dy \\ &\le C_n \|f\|_2^2 R^{n+2n\varepsilon} \\ &\le C_n R^{2na}.\end{aligned}$$

Thus,

$$\|R_j f(x)\cdot |x|^{nb}\|_2 \le C_n R^{na}$$

and

$$\begin{aligned}\mathfrak{N}(R_j f) &\equiv \|R_j f\|_2^{a/b}\cdot \|R_j f(x)\cdot |x|^{nb}\|_2^{1-a/b} \\ &\le C_n R^{n(1/2-1/p)a/b}\cdot R^{na(1-a/b)} \\ &= C_n.\end{aligned}$$

To complete the proof, it remains to show that

$$\int_{\mathbb{R}^n} R_j f(x)\cdot x^\alpha\,dx = 0 \qquad \text{for} \quad |\alpha| \le s-1.$$

We first claim $R_j f(x)\cdot x^\alpha \in L^1$. For $|\alpha| \le s-1$, since we have shown $R_j f(x)\cdot |x|^{nb} \in L^2$, we use Schwarz's inequality to get

$$\int_{|x|>1} |R_j f(x)|\cdot |x|^{|\alpha|}\,dx \le \|R_j f(x)\cdot |x|^{nb}\|_2 \left(\int_{|x|>1} |x|^{2|\alpha|-2nb}\,dx\right)^{1/2} < \infty$$

and

$$\int_{|x|\le 1} |R_j f(x)|\cdot |x|^{|\alpha|}\,dx \le \|R_j f\|_2 \left(\int_{|x|\le 1} |x|^{2|\alpha|}\,dx\right)^{1/2} < \infty.$$

We thus have $R_j f(x)\cdot x^\alpha \in L^1(\mathbb{R}^n)$ for $|\alpha| \le s-1$, and hence

$$D^\alpha (R_j f)\hat{\ }(\xi) = C(R_j f(x)\cdot x^\alpha)\hat{\ }(\xi) \qquad \text{for } |\alpha| \le s-1$$

is continuous. Moreover, it follows from [8, Lemma 9.1] that $\hat{f}$ is $(s-1)$-th order differentiable and $\hat{f}(\xi) = O(|\xi|^s)$ as $|\xi| \to 0$. We write e_j to be the j-th standard basis vector of $\mathbb{R}^n$, $\alpha = (\alpha_1, \ldots, \alpha_n)$ a multi-index of nonnegative integers α_j, $\Delta_{he_j} g(x) = g(x) - g(x - he_j)$, $\Delta_{he_j}^{\alpha_j} g = \Delta_{he_j}(\Delta_{he_j}^{\alpha_j - 1} g)$ for $\alpha_j \ge 2$, $\Delta_{he_j}^0 g = g$, and $\Delta_h^\alpha = \Delta_{he_1}^{\alpha_1}\Delta_{he_2}^{\alpha_2}\cdots\Delta_{he_n}^{\alpha_n}$. Then the boundedness of $m_j(x) = -ix_j/|x|$ implies

$$\begin{aligned}\left|\int_{\mathbb{R}^n} R_j f(x)\cdot x^\alpha\,dx\right| &= C_n \left|D^\alpha(\widehat{R_j f})(0)\right| \\ &= C_n \left|\lim_{h\to 0} |h|^{-|\alpha|}\Delta_{|h|}^\alpha (m_j \hat{f})(0)\right| \\ &\le C_n \lim_{h\to 0} |h|^{s-|\alpha|} \\ &= 0 \qquad \text{for} \quad |\alpha| \le s-1.\end{aligned}$$

Thus, the proof is finished. ■

5 The H^p Boundedness of Hilbert Transform

Following a similar but easier argument, we also have the following H^p boundedness of Hilbert transform. We leave details to readers.

Theorem 5.1. *The Hilbert transform is bounded on* $H^p(\mathbb{R})$, $0 < p \leq 1$.

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