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# A CAUCHY PROBLEM FOR ABSTRACT FRACTIONAL DIFFERENTIAL EQUATIONS WITH INFINITE DELAY

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**Abstract**

In this paper, we investigate the mild solutions of a Cauchy problem for abstract fractional differential equations with infinite delay. Several theorems of existence and uniqueness of mild solutions are established. An example is also given to illustrate our main result.

**AMS Subject Classification:** 39B52, 26A33.

**Keywords**: Fractional differential equation; infinite delay; mild solution.

## 1 Introduction

Recently, fractional differential equations have been of much interest to many researchers due to its applications in various fields, such as physics, chemistry, engineering, etc (see e.g., [10], [13] for more details). Nice results on fractional differential equations in $\mathbb{R}$ have been obtained by V. Lakshmikantham and A.S Vatsala [4] $-$[6] and other researchers (see e.g., [1, 2, 12, 15]). To the best of the author's knowledge, results for fractional differential equations in general Banach spaces even without delay are rare (cf. G.M. N'Guérékata [11]).

In this paper, we consider the following Cauchy problem for abstract fractional differential equation with infinite delay:

$$\begin{cases} D^q x(t) = f(t, x(t), x_t), & t \in [0,T] \\ x_0 = \phi \end{cases} \tag{1.1}$$

where $0 < q \leq 1$, $0 < T < \infty$, $\phi \in \mathcal{P}$, $X$ is a Banach space, $\mathcal{P}$ is a phase space and $f \in C([0,T] \times X \times \mathcal{P})$.

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After the work of J.K. Hale and J. Kato [3] and K. Schumacher [14], phase spaces have been widely used in the study of various differential equations with infinite delay. For more details, see e.g., [7] -[9].

## 2 Preliminaries

Throughout this paper $(X, \|\cdot\|)$ will be a Banach space and $x_t(\cdot) := x(t+\cdot)$ is valued in $X$. The phase space $(\mathcal{P}, \|\cdot\|_{\mathcal{P}})$ is a Banach space consisting of functions from $(-\infty, 0]$ into $X$ satisfying the following assumptions

$(\mathrm{H}_1)$ For any $t_0 \in \mathbb{R}$ and $a > 0$, if $x : (-\infty, t_0 + a] \to X$ is continuous on $[t_0, t_0 + a]$ and $x_{t_0} \in \mathcal{P}$, then $x_t \in \mathcal{P}$ and $x_t$ is continuous in $t \in [t_0, t_0 + a]$.

$(\mathrm{H}_2)$ There exist nonnegative, measurable, and locally bounded functions $K(t)$ and $M(t)$ of $t \geq 0$ such that

$$\|x_t\|_{\mathcal{P}} \leq K(t-t_0) \sup_{s\in[t_0,t]} \|x(s)\| + M(t-t_0)\|x_{t_0}\|_{\mathcal{P}}$$

for $t \in [t_0, t_0 + a]$ and $x$ as in $(\mathrm{H}_1)$.

Phase space is a classical concept in the study of functional differential equations with infinite delay. Some examples of phase spaces were given in [7].

**Definition 2.1.** *A mild solution of (1.1) is a function* $x : (-\infty, T] \to X$ *which is continuous in* $[0, T]$ *and satisfies*

$$x(t) = \begin{cases} \phi(0) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{(q-1)} f(s, x(s), x_s) ds, & t \in [0, T] \\ \phi(t), & t \in (-\infty, 0]. \end{cases} \tag{2.1}$$

## 3 Main results

We further assume that:

$(\mathrm{H}_3)$ There is a positive constant $L$ such that

$$\|f(t, u_1, v_1) - f(t, u_2, v_2)\| \leq L(\|u_1 - u_2\| + \|v_1 - v_2\|_{\mathcal{P}})$$

for all $t \in [0, T]$, $u_1, u_2 \in X$, $v_1, v_2 \in \mathcal{P}$.

$(\mathrm{H}_4)$ There is a positive function $\mu(t) \in L^2(0, T)$ such that

$$\|f(t, u_1, v_1) - f(t, u_2, v_2)\| \leq \mu(t)(\|u_1 - u_2\| + \|v_1 - v_2\|_{\mathcal{P}})$$

for all $t \in [0, T]$, $u_1, u_2 \in X$, $v_1, v_2 \in \mathcal{P}$.

$(\mathrm{H}_5)$ For every $r > 0$ there exist a constant $L(r) > 0$ such that

$$\|f(t, u_1, v_1) - f(t, u_2, v_2)\| \leq L(r)(\|u_1 - u_2\| + \|v_1 - v_2\|_{\mathcal{P}}).$$

for any $\|u_1\|, \|u_2\|, \|v_1\|_{\mathcal{P}}, \|v_2\|_{\mathcal{P}} \leq r$.

Under condition $(\mathrm{H}_3)$, we have a basic theorem.

**Theorem 3.1.** *Let $f \in C([0,T] \times X \times \mathcal{P})$ and satisfies $(\mathrm{H}_3)$. Then (1.1) has a unique mild solution.*

*Proof.* We denote $\mathcal{P}^{[0,T]} := \{x : (-\infty, T] \to X; \;\; x|_{[0,T]} \in C([0,T], X) \text{ and } x_0 \in \mathcal{P}\}$. Then, $\mathcal{P}^{[0,T]}$ is a Banach space under the norm:

$$\|x\|_{\mathcal{P}^{[0,T]}} := \sup_{t\in[0,T]} \|x(t)\| + \|x_0\|_{\mathcal{P}}.$$

For each $\phi \in \mathcal{P}$, let $\mathcal{P}_\phi^{[0,T]} := \{x \in \mathcal{P}^{[0,T]}; \;\; x_0 = \phi\}$. Clearly, $\mathcal{P}_\phi^{[0,T]}$ is a closed convex subset of $\mathcal{P}^{[0,T]}$. We define the nonlinear operator

$$(Fx)(t) = \begin{cases} \phi(0) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{(q-1)} f(s, x(s), x_s) ds, & t \in [0,T] \\ \phi(t), & t \in (-\infty, 0]. \end{cases} \tag{3.1}$$

It is easy to see that $F$ maps $\mathcal{P}_\phi^{[0,T]}$ into itself. Moreover, for $t \in [0,T]$, $x, y \in \mathcal{P}_\phi^{[0,T]}$, we have

$$\begin{aligned} \|(Fx)(t) - (Fy)(t)\| &= \|\frac{1}{\Gamma(q)} \int_0^t (t-s)^{(q-1)} [f(s,x(s),x_s) - f(s,y(s),y_s)] ds\| \\ &\leq \|\frac{1}{\Gamma(q)} \int_0^t (t-s)^{(q-1)} L[\|x(s) - y(s)\| + \|x_s - y_s\|_{\mathcal{P}}] ds \\ &\leq \frac{Lt^q}{\Gamma(q+1)} (1 + \sup_{t\in[0,T]} K(t)) \sup_{s\in[0,t]} \|x(s) - y(s)\| \end{aligned}$$

by $(\mathrm{H}_2)$.
Furthermore, by induction, we get, $k = 1, 2, \ldots$

$$\|(F^k x)(t) - (F^k y)(t)\| \leq \frac{L^k t^{kq}}{\Gamma(kq+1)} (1 + \sup_{t\in[0,T]} K(t))^k \sup_{s\in[0,t]} \|x(s) - y(s)\|.$$

Since $\lim_{k\to\infty} \frac{L^k t^{kq}}{\Gamma(kq+1)} (1 + \sup_{t\in[0,T]} K(t))^k = 0$, we can choose an integer k large enough such that $F^k$ is a contraction on $\mathcal{P}_\phi^{[0,T]}$. Therefore, by a well known extension of the contraction mapping theorem, $F$ has a unique fixed point $x \in \mathcal{P}_\phi^{[0,T]}$ which is the mild solution of (1.1). □

*Remark* 3.2. *This theorem extends the result of Theorem 3.3 in [1] even in the case that $X = \mathbb{R}$.*

If $\frac{1}{2} < q \leq 1$, we have a more general theorem.

**Theorem 3.3.** *Let $f \in C([0,T] \times X \times \mathcal{P})$ and satisfies $(\mathrm{H}_4)$. If $\frac{1}{2} < q \leq 1$, then (1.1) has a unique mild solution.*

*Proof.* Let $F$ be the operator defined above and $l = \|\mu\|_{L^2(0,T)}$. Then for any $t \in [0,T]$, $x,y \in \mathcal{P}_\phi^{[0,T]}$, we have

$$\begin{aligned}
\|(Fx)(t)-(Fy)(t)\| &= \|\frac{1}{\Gamma(q)}\int_0^t (t-s)^{(q-1)}[f(s,x(s),x_s)-f(s,y(s),y_s)]ds\| \\
&\le \|\frac{1}{\Gamma(q)}\int_0^t (t-s)^{(q-1)}\mu(s)[\|x(s)-y(s)\|+\|x_s-y_s\|_{\mathcal{P}}]ds \\
&\le \frac{1}{\Gamma(q)}(1+\sup_{t\in[0,T]}K(t))\sup_{s\in[0,t]}\|x(s)-y(s)\|\int_0^t (t-s)^{(q-1)}\mu(s)ds \\
&\le \frac{l}{\Gamma(q)}(1+\sup_{t\in[0,T]}K(t))\sup_{s\in[0,t]}\|x(s)-y(s)\|[\int_0^t (t-s)^{2(q-1)}ds]^{\frac{1}{2}} \\
&= \frac{\Gamma(2q-1)^{\frac{1}{2}}t^{\frac{2q-1}{2}}l}{\Gamma(q)\Gamma(2q)^{\frac{1}{2}}}(1+\sup_{t\in[0,T]}K(t))\sup_{s\in[0,t]}\|x(s)-y(s)\|.
\end{aligned}$$

By induction, for $k = 1,2,\ldots$, we have

$$\|(F^k x)(t)-(F^k y)(t)\| \le \frac{\Gamma(2q-1)^{\frac{k}{2}}t^{\frac{k(2q-1)}{2}}l^k}{\Gamma(q)^k\Gamma(k(2q-1)+1)^{\frac{1}{2}}}(1+\sup_{t\in[0,T]}K(t))^k\sup_{s\in[0,t]}\|x(s)-y(s)\|.$$

Standard argument shows that $F$ has a unique fixed point in $\mathcal{P}_\phi^{[0,T]}$ which is the mild solution of (1.1). □

Under condition $(\mathrm{H}_5)$, we first establish a local existence theorem.

**Theorem 3.4.** *Let $f \in C([0,T]\times X\times\mathcal{P})$ and satisfies $(\mathrm{H}_5)$. Then, there exists a real number $T' \in (0,T)$ such that (1.1) has a unique mild solution on the interval $(-\infty,T')$.*

*Proof.* Take a real number $a > 0$. We denote

$$\mathcal{P}_{\phi,a}^{[0,T]} := \{x \in \mathcal{P}_\phi^{[0,T]};\ \sup_{t\in[0,T]}\|x(t)\| \le \|\phi(0)\|+a\}.$$

Set

$$b = \sup_{t\in[0,T]}\{K(t),M(t),\|\phi(0)\|,\|\phi\|_{\mathcal{P}}\},\quad r = \max\{a+b,b(a+2b)\}$$

It is clear that, for $x \in \mathcal{P}_{\phi,a}^{[0,T]}$

$$\max_{t\in[0,T]}\{\|x(t)\|,\|x_t\|_{\mathcal{P}}\} \le r.$$

For $t \in [0,T]$ and $x \in \mathcal{P}_{\phi,a}^{[0,T]}$, we have

$$\|Fx(t)-\phi(0)\| \le \frac{t^q}{\Gamma(q+1)}(\|f(t,0,0)\|+2rL(r))$$

Moreover, for $x, y \in \mathcal{P}_{\phi,a}^{[0,T]}$ and $t \in [0,T]$

$$\|Fx(t) - Fy(t)\| \leq \frac{L(r)t^q}{\Gamma(q+1)}(1+b) \sup_{s\in[0,t]} \|x(s) - y(s)\|$$

Therefore, for any given $0 < \varepsilon < 1$, there is a real number $T' \in (0,T)$ such that for $x, y \in \mathcal{P}_{\phi,a}^{[0,T']}$

$$\max_{t\in[0,T']} \|Fx(t) - \phi(0)\| < a$$

$$\|Fx - Fy\|_{\mathcal{P}^{[0,T']}} \leq \varepsilon \|x - y\|_{\mathcal{P}^{[0,T']}}.$$

By the contraction mapping theorem, $F$ has a unique fixed point in $\mathcal{P}_{\phi,a}^{[0,T']}$, which is the mild solution of (1.1) on $(-\infty, T')$. □

**Theorem 3.5.** *Let* $f \in C([0,T] \times X \times \mathcal{P})$ *and satisfies* $(\mathrm{H}_5)$. *A mild solution of (1.1) is unique on any interval* $(-\infty, \tau] \subset (-\infty, T]$*, if it exists.*

*Proof.* Let $u, v$ be two mild solutions of (1.1) on $(-\infty, \tau]$. Set

$$t_0 = \max\{t \in [0,\tau];\ \ u|_{[0,t]} \equiv v|_{[0,t]}\}.$$

If $t_0 < \tau$, consider the equation

$$\begin{cases} D^q x(t) = f(t, x(t), x_t), & t \in [t_0, \tau] \\ x_{t_0} = u_{t_0} \end{cases} \tag{3.2}$$

Similarly as the proof of theorem (3.4), we can find a $\gamma \in (0, \tau - t_0]$ such that Eq.(3.2) has a unique solution on $[t_0, t_0 + \gamma]$ which contradicts the definition of $t_0$. Thus we have $t_0 = \tau$ and complete our proof. □

**Definition 3.6.** *A function x is a maximum mild solution of (1.1) if x is the mild solution of (1.1) on* $(-\infty, t]$*, moreover, for any y is a mild solution of (1.1) on* $(-\infty, \tau]$*, then* $\tau \leq t$*.*

Next, we extends the mild solution to the maximum one.

**Theorem 3.7.** *Let* $f \in C([0,T] \times X \times \mathcal{P})$ *and satisfies* $(\mathrm{H}_5)$. *If x is the maximum mild solution of (1.1) on* $(-\infty, T_0)$*. Then any one of the following conditions holds.*

$(\mathrm{C}_1)$ $T_0 = T$

$(\mathrm{C}_2)$ $\lim sup_{t\to T_0^-} \|x(t)\| = \infty$.

*Proof.* Assume that $T_0 < T$, we take $0 < c < T - T_0$. If $\lim sup_{t\to T_0^-} \|x(t)\| < \infty$, then there exists a constant $b_1$ such that

$$\max\{\sup_{t\in[0,T_0+c]} \{K(t), M(t)\}, \sup_{t\in[0,T_0]} \|x(t)\|, \|\phi\|_{\mathcal{P}}\} \leq b_1.$$

Set

$$\mathcal{P}_{\phi,a_1}^{[T_0,T_0+c]} = \{y \in \mathcal{P}_{\phi}^{[0,T_0+c]};\ \ y|_{(-\infty,T_0]} = x|_{(-\infty,T_0]}, \max_{t\in[T_0,T_0+c]} \|y(t)\| \leq \|x(T_0)\| + a_1\}.$$

Choose $r_1 = \max\{a_1+b_1, b_1(a_1+2b_1)\}$. Then for any $y \in \mathcal{P}_{\phi,a_1}^{[T_0,T_0+c]}$,

$$\max_{t\in[0,T_0+c]}\{\|y(t)\|, \|y_t\|_{\mathcal{P}}\} \le r_1.$$

Define the nonlinear operator

$$(Fy)(t) = \begin{cases} x(T_0) + \frac{1}{\Gamma(q)}\int_{T_0}^{T_0+c}(t-s)^{(q-1)} f(s,y(s),y_s)ds, & t\in[T_0,T_0+c] \\ x(t), & t\in(-\infty,T_0]. \end{cases} \tag{3.3}$$

Same reason as in theorem (3.4), there exists $c' \in [0,c]$ such that $F$ has a unique fixed point in $\mathcal{P}_{\phi,a_1}^{[T_0,T_0+c']}$. This fixed point will extends the mild solution $x$ to $(-\infty, T_0+c']$ which contradicts the definition of $T_0$. □

# 4 An example

To illustrate the usefulness of our main result, we consider the following Cauchy problem

$$\begin{cases} D^q x(t) = \dfrac{Lt^2}{1+t^2}(x(t)+x_t(-\dfrac{1}{2}t)), & t\in[0,T] \\ x_0 = \phi \end{cases} \tag{4.1}$$

where $L>0$ and $\phi \in \mathcal{P} := \{\phi(\theta);\ \ \phi(\theta)$ is bounded uniformly continuous function from $(-\infty,0]$ to $\mathbb{X}\}$ with the norm

$$\|\phi(\cdot)\|_{\mathcal{P}} = \sup_{\theta\in(-\infty,0]}\|\phi(\theta)\|.$$

Clearly, $\mathcal{P}$ is a phase space (see [7]). Set

$$f(t,u,v) = \frac{Lt^2}{1+t^2}(u+v(-\frac{1}{2}t)),$$

where $(t,u,v)\in[0,T]\times\mathbb{X}\times\mathcal{P}$. Then, we have

$$\begin{aligned} \|f(t,u_1,v_1)-f(t,u_2,v_2)\| &\le \frac{Lt^2}{1+t^2}(\|u_1-u_2\|+\|v_1(-\frac{1}{2}t)-v_2(-\frac{1}{2}t)\|) \\ &\le L(\|u_1-u_2\|+\|v_1-v_2\|_{\mathcal{P}}), \end{aligned}$$

which means that $(\mathrm{H}_3)$ holds. By theorem (3.1), Eq.(4.1) has a unique mild solution on $(-\infty,T]$.


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